- **関係式**

$$\eta^{\mu\nu}l_\mu l_\nu = 0$$

$$l^\alpha l_{\alpha|\beta} = 0$$

$$v_\alpha = l^\mu l_{\alpha|\mu} = -Al_\alpha$$

$$v_{\mu|\nu} = l^\alpha_{|\nu} l_{\mu|\alpha} + l^\alpha l_{\mu|\alpha|\nu} = -A_{|\nu} l_\mu - Al_{\mu|\nu}$$

- **m^3 の式**

$$l^\beta l^\lambda \eta^{\alpha\sigma}[\beta\mu,\sigma][\alpha\nu,\lambda] + l^\alpha l^\sigma \eta^{\beta\lambda}[\beta\mu,\sigma][\alpha\nu,\lambda] = 0$$

第一項は

$$\begin{aligned}
l^\beta l^\lambda \eta^{\alpha\sigma}[\beta\mu,\sigma][\alpha\nu,\lambda] &= m^2 l^\beta l^\lambda \eta^{\alpha\sigma}\big\{(l_\mu l_\sigma)_{|\beta}(l_\nu l_\lambda)_{|\alpha} + (l_\mu l_\sigma)_{|\beta}(l_\lambda l_\alpha)_{|\nu} - (l_\mu l_\sigma)_{|\beta}(l_\alpha l_\nu)_{|\lambda} \\
&\qquad + (l_\sigma l_\beta)_{|\mu}(l_\nu l_\lambda)_{|\alpha} + (l_\sigma l_\beta)_{|\mu}(l_\lambda l_\alpha)_{|\nu} - (l_\sigma l_\beta)_{|\mu}(l_\alpha l_\nu)_{|\lambda} \\
&\qquad - (l_\beta l_\mu)_{|\sigma}(l_\nu l_\lambda)_{|\alpha} - (l_\beta l_\mu)_{|\sigma}(l_\lambda l_\alpha)_{|\nu} + (l_\beta l_\mu)_{|\sigma}(l_\alpha l_\nu)_{|\lambda}\big\} \\
&= m^2 l^\beta l^\lambda \eta^{\alpha\sigma}\big(-(l_\mu l_\sigma)_{|\beta}(l_\alpha l_\nu)_{|\lambda}\big) \\
&= m^2 l^\beta l^\lambda \big(-(l_{\mu|\beta} l^\alpha + l_\mu l^\alpha_{|\beta})(l_{\alpha|\lambda} l_\nu + l_\alpha l_{\nu|\lambda})\big) \\
&= m^2 l_\mu l_\nu (-l^\beta l^\alpha_{|\beta} l_{\alpha|\lambda} l^\lambda)
\end{aligned}$$

第二項は

$$\begin{aligned}
l^\alpha l^\lambda \eta^{\beta\sigma}[\beta\mu,\lambda][\alpha\nu,\sigma] &= m^2 l^\alpha l^\lambda \eta^{\beta\sigma}\big((l_\mu l_\lambda)_{|\beta}(l_\nu l_\sigma)_{|\alpha} + (l_\mu l_\lambda)_{|\beta}(l_\sigma l_\alpha)_{|\nu} - (l_\mu l_\lambda)_{|\beta}(l_\alpha l_\nu)_{|\sigma} \\
&\qquad + (l_\lambda l_\beta)_{|\mu}(l_\nu l_\lambda)_{|\alpha} + (l_\lambda l_\beta)_{|\mu}(l_\lambda l_\alpha)_{|\nu} - (l_\lambda l_\beta)_{|\mu}(l_\alpha l_\nu)_{|\sigma} \\
&\qquad - (l_\beta l_\mu)_{|\lambda}(l_\nu l_\sigma)_{|\alpha} - (l_\beta l_\mu)_{|\lambda}(l_\sigma l_\alpha)_{|\nu} + (l_\beta l_\mu)_{|\lambda}(l_\alpha l_\nu)_{|\sigma}\big) \\
&= m^2 l^\alpha l^\lambda \eta^{\beta\sigma}\big(-(l_\beta l_\mu)_{|\lambda}(l_\nu l_\sigma)_{|\alpha}\big) \\
&= m^2 l^\alpha l^\lambda \eta^{\beta\sigma}\big(-(l_{\beta|\lambda} l_\mu + l_\beta l_{\mu|\lambda})(l_{\nu|\alpha} l_\sigma + l_\nu l_{\sigma|\alpha})\big) \\
&= m^2 l_\mu l_\nu \big(-(l_\beta l^\alpha)_{|\lambda}(l^\lambda l^\beta)_{|\alpha}\big) \\
&= m^2 l_\mu l_\nu \big(-(l_{\beta|\lambda} l^\alpha + l_\beta l^\alpha_{|\lambda})(l^\lambda_{|\alpha} l^\beta + l^\lambda l^\beta_{|\alpha})\big) \\
&= -m^2 l_\mu l_\nu l_{\beta|\lambda} l^\alpha l^\lambda l^\beta_{|\alpha}
\end{aligned}$$

第一項と第二項を合わせて、m^2 を落とすと

$$-l_\mu l_\nu(v^\alpha v_\alpha)=0 \quad (v^\alpha = l^\beta l^\alpha_{||\beta} = l^\beta l^\alpha_{|\beta})$$

- **m^2 の式**

$$2m(l^\alpha l^\rho[\mu\nu,\rho])_{|\alpha} - \eta^{\alpha\sigma}\eta^{\beta\lambda}[\beta\mu,\sigma][\alpha\nu,\lambda]=0$$

第一項は

$$2m(l^\alpha l^\rho[\mu\nu,\rho])_{|\alpha} = 2m\big((l^\alpha l^\rho)_{|\alpha}[\mu\nu,\rho] + l^\alpha l^\rho[\mu\nu,\rho]_{|\alpha}\big)$$

これの右辺の第一種クリストッフェル記号は

$$\begin{aligned}
[\mu\nu,\rho] &= m\big(-(l_\nu l_\rho)_{|\mu} - (l_\rho l_\mu)_{|\nu} + (l_\mu l_\nu)_{|\rho}\big)\\
[\mu\nu,\rho]_{|\alpha} &= m\big(-(l_\nu l_\rho)_{|\mu} - (l_\rho l_\mu)_{|\nu} + (l_\mu l_\nu)_{|\rho}\big)_{|\alpha}
\end{aligned}$$

右辺の第一項は

$$\begin{aligned}
2m(l^\alpha l^\rho)_{|\alpha}[\mu\nu,\rho] =& -2m^2(l^\alpha_{|\alpha}l^\rho + l^\alpha l^\rho_{|\alpha})((l_\nu l_\rho)_{|\mu} + (l_\rho l_\mu)_{|\nu} - (l_\mu l_\nu)_{|\rho})\\
=& -2m^2(-l^\alpha_{|\alpha}l^\rho(l_\mu l_\nu)_{|\rho} + l^\alpha l^\rho_{|\alpha}(l_\nu l_\rho)_{|\mu} + l^\alpha l^\rho_{|\alpha}(l_\rho l_\mu)_{|\nu} - l^\alpha l^\rho_{|\alpha}(l_\mu l_\nu)_{|\rho})\\
=& -2m^2(-l^\alpha_{|\alpha}l^\rho(l_{\mu|\rho}l_\nu + l_\mu l_{\nu|\rho}) + v^\rho(l_{\nu|\mu}l_\rho + l_\nu l_{\rho|\mu})\\
&+ v^\rho(l_{\rho|\nu}l_\mu + l_\rho l_{\mu|\nu})_{|\nu} - v^\rho(l_{\mu|\rho}l_\nu + l_\mu l_{\nu|\rho}))\\
=& -2m^2(-l^\alpha_{|\alpha}l^\rho(l_{\mu|\rho}l_\nu + l_\mu l_{\nu|\rho}) - Al^\rho(l_{\nu|\mu}l_\rho + l_\nu l_{\rho|\mu})\\
&- Al^\rho(l_{\rho|\nu}l_\mu + l_\rho l_{\mu|\nu})_{|\nu} + Al^\rho(l_{\mu|\rho}l_\nu + l_\mu l_{\nu|\rho}))\\
=& -2m^2(-l^\alpha_{|\alpha}(v_\mu l_\nu + l_\mu v_\nu) + A(v_\mu l_\nu + v_\nu l_\mu))\\
=& -2m^2(-l^\alpha_{|\alpha}(-Al_\mu l_\nu - Al_\mu l_\nu) + A(-Al_\mu l_\nu - Al_\mu l_\nu))\\
=& -2m^2(l^\alpha_{|\alpha}Al_\mu l_\nu + l^\alpha_{|\alpha}Al_\mu l_\nu - A^2 l_\mu l_\nu - A^2 l_\mu l_\nu)\\
=& -2m^2 l_\mu l_\nu(2l^\alpha_{|\alpha}A - 2A^2)
\end{aligned}$$

第二項は

$$
\begin{aligned}
2ml^\alpha l^\rho[\mu\nu,\rho]_{|\alpha} =& -2m^2 l^\alpha l^\rho(((l_{\nu|\mu}l_\rho + l_\nu l_{\rho|\mu}) + (l_{\rho|\nu}l_\mu + l_\rho l_{\mu|\nu}) - (l_{\mu|\rho}l_\nu + l_\mu l_{\nu|\rho}))_{|\alpha}) \\
=& -2m^2 l^\alpha l^\rho((l_{\nu|\mu|\alpha}l_\rho + l_{\nu|\mu}l_{\rho|\alpha} + l_{\nu|\alpha}l_{\rho|\mu} + l_\nu l_{\rho|\mu|\alpha}) \\
& + (l_{\rho|\nu|\alpha}l_\mu + l_{\rho|\nu}l_{\mu|\alpha} + l_{\rho|\alpha}l_{\mu|\nu} + l_\rho l_{\mu|\nu|\alpha}) - (l_{\mu|\rho|\alpha}l_\nu + l_{\mu|\rho}l_{\nu|\alpha} + l_{\mu|\alpha}l_{\nu|\rho} + l_\mu l_{\nu|\rho|\alpha})) \\
=& -2m^2(l^\alpha l^\rho l_\nu l_{\rho|\mu|\alpha} + l^\alpha l^\rho l_\mu l_{\rho|\nu|\alpha} - l^\alpha l^\rho l_\nu l_{\mu|\rho|\alpha} - l^\alpha l^\rho l_{\mu|\rho}l_{\nu|\alpha} - l^\alpha l^\rho l_{\mu|\alpha}l_{\nu|\rho} - l^\alpha l^\rho l_\mu l_{\nu|\rho|\alpha})
\end{aligned}
$$

v_ρ の微分より

$$
\begin{aligned}
v_{\rho|\mu} &= l^\alpha_{|\mu} l_{\rho|\alpha} + l^\alpha l_{\rho|\alpha|\mu} = -A_{|\mu}l_\rho - Al_{\rho|\mu} \\
l^\rho v_{\rho|\mu} &= -A_{|\mu}l^\rho l_\rho - Al^\rho l_{\rho|\mu} = 0 \\
l^\rho l_\nu l^\alpha l_{\rho|\mu|\alpha} &= l^\rho l_\nu l^\alpha l_{\rho|\alpha|\mu} = l^\rho l_\nu (v_{\rho|\mu} - l^\alpha_{|\mu} l_{\rho|\alpha})
\end{aligned}
$$

これらを使って

$$
\begin{aligned}
2ml^\alpha l^\rho[\mu\nu,\rho]_{|\alpha} =& -2m^2(l^\rho l_\nu(v_{\rho|\mu} - l^\alpha_{|\mu}l_{\rho|\alpha}) + l^\rho l_\mu(v_{\rho|\nu} - l^\alpha_{|\nu}l_{\rho|\alpha}) - l^\rho l_\nu(v_{\mu|\rho} - l^\alpha_{|\rho}l_{\mu|\alpha}) \\
& \qquad - v_\mu v_\nu - v_\mu v_\nu - l^\rho l_\mu(v_{\nu|\rho} - l^\alpha_{|\rho}l_{\nu|\alpha})) \\
=& -2m^2(-l^\rho l_\nu(v_{\mu|\rho} - l^\alpha_{|\rho}l_{\mu|\alpha}) - l^\rho l_\mu(v_{\nu|\rho} - l^\alpha_{|\rho}l_{\nu|\alpha}) - 2v_\mu v_\nu) \\
=& -2m^2(-l^\rho l_\nu(-Al_{\mu|\rho} - A_{|\rho}l_\mu - l^\alpha_{|\rho}l_{\mu|\alpha}) - l^\rho l_\mu(-Al_{\nu|\rho} - A_{|\rho}l_\nu - l^\alpha_{|\rho}l_{\nu|\alpha}) - 2v_\mu v_\nu) \\
=& -2m^2(Al^\rho l_\nu l_{\mu|\rho} + A_{|\rho}l^\rho l_\nu l_\mu + l^\rho l_\nu l^\alpha_{|\rho}l_{\mu|\alpha} + Al^\rho l_\mu l_{\nu|\rho} + A_{|\rho}l^\rho l_\mu l_\nu + l^\rho l_\mu l^\alpha_{|\rho}l_{\nu|\alpha} - 2A^2 l_\mu l_\nu) \\
=& -2m^2(Al_\nu l^\rho l_{\mu|\rho} + A_{|\rho}l^\rho l_\mu l_\nu - Al_\nu l^\alpha l_{\mu|\alpha} + Al_\mu l^\rho l_{\nu|\rho} + A_{|\rho}l^\rho l_\mu l_\nu - Al_\mu l^\alpha l_{\nu|\alpha} - 2A^2 l_\mu l_\nu) \\
=& -2m^2(-A^2 l_\mu l_\nu + A_{|\rho}l^\rho l_\nu l_\mu + A^2 l_\mu l_\nu - A^2 l_\nu l_\mu + A_{|\rho}l^\rho l_\mu l_\nu + A^2 l_\nu l_\mu - 2A^2 l_\mu l_\nu) \\
=& -2m^2 l_\mu l_\nu(2A_{|\rho}l^\rho - 2A^2)
\end{aligned}
$$

m^2 の式の第二項は

$$
\begin{aligned}
\eta^{\alpha\sigma}\eta^{\beta\lambda}[\beta\mu,\sigma][\alpha\nu,\lambda] &= m^2\eta^{\alpha\sigma}\eta^{\beta\lambda}[(l_\mu l_\sigma)_{|\beta}(l_\nu l_\lambda)_{|\alpha} + (l_\mu l_\sigma)_{|\beta}(l_\lambda l_\alpha)_{|\nu} - (l_\mu l_\sigma)_{|\beta}(l_\alpha l_\nu)_{|\lambda} \\
&\qquad + (l_\sigma l_\beta)_{|\mu}(l_\nu l_\lambda)_{|\alpha} + (l_\sigma l_\beta)_{|\mu}(l_\lambda l_\alpha)_{|\nu} - (l_\sigma l_\beta)_{|\mu}(l_\alpha l_\nu)_{|\lambda} \\
&\qquad - (l_\beta l_\mu)_{|\sigma}(l_\nu l_\lambda)_{|\alpha} - (l_\beta l_\mu)_{|\sigma}(l_\lambda l_\alpha)_{|\nu} + (l_\beta l_\mu)_{|\sigma}(l_\alpha l_\nu)_{|\lambda}] \\
&= m^2\eta^{\alpha\sigma}\eta^{\beta\lambda}((l_\mu l_\sigma)_{|\beta}(l_\nu l_\lambda)_{|\alpha} + l_\mu l_{\sigma|\beta} l_\lambda l_{\alpha|\nu} - l_\mu l_{\sigma|\beta} l_\nu l_{\alpha|\lambda} \\
&\qquad + l_\sigma l_{\beta|\mu} l_\nu l_{\lambda|\alpha} - l_{\sigma|\mu} l_\beta l_{\alpha|\lambda} l_\nu \\
&\qquad - l_{\beta|\sigma} l_\mu l_\nu l_{\lambda|\alpha} - l_{\beta|\sigma} l_\mu l_{\lambda|\nu} l_\alpha + (l_\beta l_\mu)_{|\sigma}(l_\alpha l_\nu)_{|\lambda}) \\
&= m^2\eta^{\alpha\sigma}\eta^{\beta\lambda}((l_\mu l_\sigma)_{|\beta}(l_\nu l_\lambda)_{|\alpha} + (l_\beta l_\mu)_{|\sigma}(l_\alpha l_\nu)_{|\lambda} - l_\mu l_{\sigma|\beta} l_\nu l_{\alpha|\lambda} - l_{\beta|\sigma} l_\mu l_\nu l_{\lambda|\alpha} \\
&\qquad + l_\mu l_{\alpha|\nu} v^\alpha + l_{\beta|\mu} l_\nu v^\beta - l^\alpha_{|\mu} l_\nu v_\alpha - l_\mu l^\beta_{|\nu} v_\beta) \\
&= m^2\eta^{\alpha\sigma}\eta^{\beta\lambda}[(l_{\mu|\beta} l_\sigma l_{\nu|\alpha} l_\lambda) + (l_{\mu|\beta} l_\sigma l_\nu l_{\lambda|\alpha}) + (l_\mu l_{\sigma|\beta} l_{\nu|\alpha} l_\lambda) + (l_\mu l_{\sigma|\beta} l_\nu l_{\lambda|\alpha}) \\
&\quad + (l_{\beta|\sigma} l_\mu l_{\alpha|\lambda} l_\nu) + (l_{\beta|\sigma} l_\mu l_\alpha l_{\nu|\lambda}) + (l_\beta l_{\mu|\sigma} l_{\alpha|\lambda} l_\nu) + (l_\beta l_{\mu|\sigma} l_\alpha l_{\nu|\lambda}) - l_\mu l_{\sigma|\beta} l_\nu l_{\alpha|\lambda} - l_{\beta|\sigma} l_\mu l_\nu l_{\lambda|\alpha}] \\
&= m^2[v_\nu v_\mu + \eta^{\beta\lambda}(l_{\mu|\beta} l_\nu v_\lambda) + \eta^{\alpha\sigma}(l_\mu l_{\nu|\alpha} v_\sigma) + \eta^{\alpha\sigma}\eta^{\beta\lambda}(l_\mu l_{\sigma|\beta} l_\nu l_{\lambda|\alpha}) \\
&\quad + \eta^{\alpha\sigma}\eta^{\beta\lambda}(l_{\beta|\sigma} l_\mu l_{\alpha|\lambda} l_\nu) + \eta^{\beta\lambda}(l_\mu l_{\nu|\lambda} v_\beta) + \eta^{\alpha\sigma}(l_{\mu|\sigma} l_\nu v_\alpha) + (v_\nu v_\mu) - l_\mu l^\alpha_{|\beta} l_\nu l^{|\beta}_\alpha - l^{|\alpha}_\beta l_\mu l_\nu l^\beta_{|\alpha}] \\
&= m^2[2A^2 l_\mu l_\nu + (-A v_\mu l_\nu) + (-A v_\nu l_\mu) + \eta^{\alpha\sigma}\eta^{\beta\lambda}(l_\mu l_{\sigma|\beta} l_\nu l_{\lambda|\alpha}) \\
&\qquad + \eta^{\alpha\sigma}\eta^{\beta\lambda}(l_{\beta|\sigma} l_\mu l_{\alpha|\lambda} l_\nu) + (-A v_\nu l_\mu) + (-A v_\mu l_\nu) - l_\mu l^\alpha_{|\beta} l_\nu l^{|\beta}_\alpha - l^{|\alpha}_\beta l_\mu l_\nu l^\beta_{|\alpha}] \\
&= m^2[2A^2 l_\mu l_\nu + (A^2 l_\mu l_\nu) + (A^2 l_\mu l_\nu) + \eta^{\alpha\sigma}\eta^{\beta\lambda}(l_\mu l_{\sigma|\beta} l_\nu l_{\lambda|\alpha}) \\
&\qquad + \eta^{\alpha\sigma}\eta^{\beta\lambda}(l_{\beta|\sigma} l_\mu l_{\alpha|\lambda} l_\nu) + (A^2 l_\mu l_\nu) + (A^2 l_\mu l_\nu) - l_\mu l^\alpha_{|\beta} l_\nu l^{|\beta}_\alpha - l^{|\alpha}_\beta l_\mu l_\nu l^\beta_{|\alpha}] \\
&= m^2 l_\mu l_\nu(2A^2 + A^2 + A^2 + \eta^{\alpha\sigma}\eta^{\beta\lambda}(l_{\sigma|\beta} l_{\lambda|\alpha}) + \eta^{\alpha\sigma}\eta^{\beta\lambda}(l_{\beta|\sigma} l_{\alpha|\lambda}) + 2A^2 - l_\mu l^\alpha_{|\beta} l_\nu l^{|\beta}_\alpha - l^{|\alpha}_\beta l_\mu l_\nu l^\beta_{|\alpha}) \\
&= m^2 l_\mu l_\nu(6A^2 + \eta^{\alpha\sigma}\eta^{\beta\lambda}(l_{\sigma|\beta} l_{\lambda|\alpha}) + \eta^{\alpha\sigma}\eta^{\beta\lambda}(l_{\beta|\sigma} l_{\alpha|\lambda}) - l^\alpha_{|\beta} l^{|\beta}_\alpha - l^{|\alpha}_\beta l^\beta_{|\alpha}) \\
&= m^2 l_\mu l_\nu(6A^2 + (l^\alpha_{|\beta} l^\beta_{|\alpha}) + (l^\lambda_{|\sigma} l^\sigma_{|\lambda}) - l^\alpha_{|\beta} l^{|\beta}_\alpha - l^{|\alpha}_\beta l^\beta_{|\alpha})
\end{aligned}
$$

よって、m^2 の式は

$$
\begin{aligned}
&-m^2 l_\mu l_\nu(4l^\alpha_{|\alpha} A - 4A^2 + 2l^\rho A_{|\rho} + 2l^\rho A_{|\rho} - 4A^2) - m^2 l_\mu l_\nu(6A^2 + (l^\alpha_{|\beta} l^\beta_{|\alpha}) + (l^\lambda_{|\sigma} l^\sigma_{|\lambda}) - l^\alpha_{|\beta} l^{|\beta}_\alpha - l^{|\alpha}_\beta l^\beta_{|\alpha}) \\
&= -m^2 l_\mu l_\nu(4(l^\alpha A)_{|\alpha} - 8A^2) - m^2 l_\mu l_\nu(6A^2 + (l^\alpha_{|\beta} l^\beta_{|\alpha}) + (l^\lambda_{|\sigma} l^\sigma_{|\lambda}) - l^\alpha_{|\beta} l^{|\beta}_\alpha - l^{|\alpha}_\beta l^\beta_{|\alpha}) \\
&= -m^2 l_\mu l_\nu(4(l^\alpha A)_{|\alpha} - 8A^2 + 6A^2 + (l^\alpha_{|\beta} l^\beta_{|\alpha}) + (l^\lambda_{|\sigma} l^\sigma_{|\lambda}) - l^\alpha_{|\beta} l^{|\beta}_\alpha - l^{|\alpha}_\beta l^\beta_{|\alpha}) \\
&= -m^2 l_\mu l_\nu(4(l^\alpha A)_{|\alpha} - 2A^2 + 2l^\alpha_{|\beta} l^\beta_{|\alpha} - 2l^\alpha_{|\beta} l^{|\beta}_\alpha)
\end{aligned}
$$